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2007 J. Phys. A: Math. Theor. 40 11361

(http://iopscience.iop.org/1751-8121/40/37/012)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 11361–11371 doi:10.1088/1751-8113/40/37/012

Quantumness of bipartite states in terms of conditional entropies

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Received 23 January 2007, in final form 12 July 2007
Published 29 August 2007
Online at stacks.iop.org/JPhysA/40/11361

Abstract

Quantum discord, as defined by Olliver and Zurek (2002 *Phys. Rev. Lett.* **88** 017901) as the difference of two natural quantum extensions of the classical mutual information, plays an interesting role in characterizing quantumness of correlations. Inspired by this idea, we will study quantumness of bipartite states arising from different quantum analogs of the classical conditional entropy. Our approach is intrinsic, in contrast to the Olliver–Zurek method that involves extrinsic local measurements. For this purpose, we introduce two alternative variants of quantum conditional entropies via conditional density operators, which in turn are intuitive quantum extensions of equivalent classical expressions for the conditional probability. The significance of these quantum conditional entropies in characterizing quantumness of bipartite states is illustrated through several examples.

PACS numbers: 03.67.−a, 89.70.+c

1. Introduction

It is well known that the process of quantization usually leads to many different (rather than a unique) quantum extensions of a classical object. This is often related to the non-commutativity of operators which represent quantum states and observables. In particular, classically equivalent expressions may have different quantum analogs, and the differences among these analogs can be used to characterize the 'quantumness' of an object. By exploiting this idea, Olliver and Zurek proposed the notion of quantum discord, which is defined as the difference of two natural quantum extensions of the classical mutual information, and demonstrated its implications for exhibiting the quantum nature of correlations in bipartite states [1, 2]. This is further explored by Herbut [3, 4]. A closely related quantity has also been introduced by Henderson and Vedral in their study of correlations in quantum states [5].

More specifically, to put their setup in a proper perspective and to motivate our investigation, let us first recall the classical notions of conditional probability, the Shannon entropy, the mutual information and conditional entropy, and their quantum analogs. Consider a classical bipartite state for a system consisting of subsystems a and b, which is mathematically represented by a joint probability distribution $p_{ab}(i, j)$ with reduced probabilities (marginals) $p_a(i) = \sum_j p_{ab}(i, j)$ and $p_b(j) = \sum_i p_{ab}(i, j)$. The conditional probability distribution given the marginal p_b is

$$p_{a|b}(i|j) = \frac{p_{ab}(i, j)}{p_b(j)}. \tag{1}$$

Clearly, for any fixed j, $p_{a|b}(i|j)$ is a probability distribution since $\sum_i p_{a|b}(i|j) = 1$. In particular, each $p_{a|b}(i|j)$ ranges from 0 to 1.

Let $H(\cdot)$ be the Shannon entropy functional, e.g., $H(p_{ab}) = -\sum_{ij} p_{ab}(i, j) \ln p_{ab}(i, j)$ (the logarithm is taken to the natural base e), then the conditional Shannon entropy given p_b is defined as

$$H(p_{ab}|p_b) = H(p_{ab}) - H(p_b), \tag{2}$$

which can be equivalently written in terms of the conditional probability (1) as

$$H(p_{ab}|p_a) = -\sum_{ij} p_{ab}(i, j) \ln p_{a|b}(i|j). \tag{3}$$

Now the classical mutual information $I(p_{ab})$ can be written as

$$\begin{aligned} I(p_{ab}) &= H(p_a) + H(p_b) - H(p_{ab}) \\ &= H(p_a) - H(p_{ab}|p_b). \end{aligned} \tag{4}$$

Passing to the quantum scenario, the states are represented by density operators acting on a composite Hilbert space and the summation is replaced by the trace. Thus a bipartite density operator ρ_{ab} plays the role of a joint probability p_{ab}, with the reduced density operator $\rho_b = \mathrm{tr}_a \rho_{ab}$ (partial trace) playing the role of the marginal probability p_b. The quantum analog of the Shannon entropy is the quantum entropy (von Neumann entropy). Let $S(\cdot)$ denote the quantum entropy functional, e.g., $S(\rho_{ab}) = -\mathrm{tr}\rho_{ab} \ln \rho_{ab}$, the corresponding quantum conditional entropy is usually defined as

$$S(\rho_{ab}|\rho_b) = S(\rho_{ab}) - S(\rho_b), \tag{5}$$

and the quantum mutual information $\mathcal{I}(\rho_{ab})$ is defined as

$$\begin{aligned} \mathcal{I}(\rho_{ab}) &= S(\rho_a) + S(\rho_b) - S(\rho_{ab}) \\ &= S(\rho_a) - S(\rho_{ab}|\rho_b), \end{aligned} \tag{6}$$

which is a direct generalization of the classical expression (4). Apart from $\mathcal{I}(\rho_{ab})$, Olliver and Zurek also considered another quantum extension of the classical mutual information based on a local measurement $\{B_j\}$ (a complete set of one-dimensional projectors) on subsystem b. The quantum state, conditioned on the measurement outcome being j, changes to

$$\rho_{ab}(j) = \frac{1}{p_j}(\mathbf{1}_a \otimes B_j)\rho_{ab}(\mathbf{1}_a \otimes B_j) \tag{7}$$

with probability $p_j = \mathrm{tr}(\mathbf{1}_a \otimes B_j)\rho_{ab}(\mathbf{1}_a \otimes B_j)$. Here $\mathbf{1}_a$ stands for the identity operator for subsystem a. They then defined an alternative form of the quantum conditional entropy given the measurement $\{B_j\}$ as

$$S(\rho_{ab}|\{B_j\}) = \sum_j p_j S(\rho_{ab}(j)), \tag{8}$$

and introduced a new variant of quantum mutual information (based on the measurement $\{B_j\}$) as

$$\mathcal{I}(\rho_{ab}|\{B_j\}) = S(\rho_a) - S(\rho_{ab}|\{B_j\}) \tag{9}$$

which is also clearly motivated by equation (4). They called the difference

$$\mathcal{D}(\rho_{ab}|\{B_j\}) = \mathcal{I}(\rho_{ab}) - \mathcal{I}(\rho_{ab}|\{B_j\}) \tag{10}$$

the quantum discord. Clearly, this quantity can be rewritten as

$$\mathcal{D}(\rho_{ab}|\{B_j\}) = S(\rho_{ab}|\{B_j\}) - S(\rho_{ab}|\rho_b)$$

which is the difference of the quantum conditional entropies defined by equations (5) and (8), respectively. This quantum discord depends on the local measurement $\{B_j\}$, which is not intrinsic to the quantum state ρ_{ab}.

Due to the subtle nature of quantumness, no single quantity can capture all its essential features, and it is desirable to characterize it in as many distinct ways as possible. In this paper, we will study the phenomena of quantum discord from a new perspective and introduce two alternative variants of quantum conditional entropies. The key idea is to introduce conditional density operators, different from $\rho_{ab}(j)$, which are intrinsic to the quantum state itself and are independent of any auxiliary measurement. Then we introduce two measures of quantumness of bipartite states by considering the differences of the various quantum conditional entropies. These measures quantify the non-commutativity between the reduced state and the whole state, and shed different and complementary insights into the nature of quantumness of the bipartite state.

The rest of the paper is arranged as follows. In section 2, we introduce several quantum extensions of conditional probability on a heuristic ground and investigate their relations. We also indicate their intrinsic implications for the reduction criterion for separability of bipartite quantum states. We compare the quantum conditional entropies based on the various conditional density operators in section 3, and investigate their applications for characterizing quantumness of bipartite states in section 4. We also illustrate the quantumness measures based on the different quantum generalizations of conditional entropy through several examples. Finally, in section 5 we present the conclusion and some discussion. It must be emphasized that our measures of quantumness are similar to the notion of quantum discord of Olliver–Zurek only formally and spiritly, the informational meaning is quite different: the quantum discord is a measure of quantumness of correlations, while our measures quantify certain aspects of quantumness of bipartite states with respect to their marginals. In this paper, we always work in finite-dimensional system spaces.

2. Conditional density operators

When we pursue quantum analogs of the conditional probability defined by equation (1) and the conditional Shannon entropy defined by equation (2), we come across the difficulty in defining a unique quantum variant, just as usually happens in the process of quantization. This ambiguity in defining a unique quantum extension of a classical quantity was exploited by Olliver and Zurek with the purpose of quantifying quantumness of correlations [1]. Given this non-uniqueness, it is of interest to make as many natural and meaningful quantum extensions of classically equivalent expressions as possible, and study their difference in revealing the quantum nature of the involved object. In this section, we will introduce several quantum extensions of conditional probability, which will be used to define various quantum conditional entropies in the following section.

First, we note that the operator $\rho_{ab}(j)$ defined by equation (7) is sometimes called the conditional density operator [6, 7]. However, it depends on an extrinsic measurement, and is not a direct quantum extension of the classical notion of conditional probability as defined by equation (1).

An elegant and intrinsic quantum extension of conditional probability is proposed by Adami and Cerf as [8]

$$\rho_{a|b} = \lim_{n\to\infty} \left(\rho_{ab}^{\frac{1}{n}}(\mathbf{1}_a \otimes \rho_b)^{-\frac{1}{n}}\right)^n = \mathrm{e}^{\ln\rho_{ab} - \ln(\mathbf{1}_a\otimes\rho_b)}, \tag{11}$$

which is well defined on the support of ρ_{ab}. The above last identity follows from the Lie-Trotter product formula. The unique significance of this conditional density operator is that if we use it to mimic the classical expression (3) to define a quantum conditional entropy $-\mathrm{tr}\rho_{ab}\ln\rho_{a|b}$, then this is equivalent to the conventional quantum conditional entropy defined by equation (5), that is,

$$S(\rho_{ab}|\rho_b) = -\mathrm{tr}\rho_{ab}\ln\rho_{a|b}. \tag{12}$$

Two drawbacks of $\rho_{a|b}$ are that it is difficult to compute, and in general $\mathrm{tr}\rho_{a|b} \neq d$, in contrast to the identity $\sum_{i,j} p_{a|b}(i|j) = d$ for the classical conditional probability. Here d is the dimension of subsystem b in the quantum case and the cardinality of the set $\{j\}$ in the classical case.

Motivated by the classical expression of conditional probability (1), one may naively define $\rho_{ab}\left(\mathbf{1}_a \otimes \rho_b^{-1}\right)$ as a variant of the quantum conditional density, but in general, ρ_{ab} and $\mathbf{1}_a \otimes \rho_b^{-1}$ do not commute, and the resulting operator may fail to be Hermitian. This can be readily remedied by symmetrization. Thus if we recast equation (1) into the equivalent forms

$$p_{a|b} = p_{ab}^{1/2} p_b^{-1} p_{ab}^{1/2} = p_b^{-1/2} p_{ab} p_b^{-1/2},$$

and pursue formal quantum analogs, then we come to two simple quantum extensions of conditional probability as

$$\rho_{a|b}^{-} = \rho_{ab}^{1/2}\left(\mathbf{1}_a \otimes \rho_b^{-1}\right)\rho_{ab}^{1/2} \tag{13}$$

and

$$\rho_{a|b}^{+} = \left(\mathbf{1}_a \otimes \rho_b^{-1/2}\right)\rho_{ab}\left(\mathbf{1}_a \otimes \rho_b^{-1/2}\right). \tag{14}$$

The signs $-$ and $+$ anticipate a latter relationship. Note that the above operator inverse is taking in a generalized sense when the relevant operator is degenerate (not invertible). To avoid unnecessary complications, we will assume henceforth that the reduced density operator ρ_b is invertible. We now have three quantum extensions of the classical conditional probabilities, i.e., $\rho_{a|b}$, $\rho_{a|b}^{-}$ and $\rho_{a|b}^{+}$ defined by equations (11), (13) and (14), respectively.

The conditional density operators $\rho_{a|b}^{-}$ and $\rho_{a|b}^{+}$, though naively defined and innocent looking, actually shed interesting light on the celebrated reduction criterion for separability of bipartite quantum states as proposed by Adami *et al* and by Horodecki [8–12]. Recall that a bipartite quantum state ρ_{ab} is separable (classically correlated) if it can be written as [13]

$$\rho_{ab} = \sum_\mu \lambda_\mu \rho_a^{(\mu)} \otimes \rho_b^{(\mu)},$$

where $\sum_\mu \lambda_\mu = 1$, $\lambda_\mu \geqslant 0$, and $\rho_a^{(\mu)}$ and $\rho_b^{(\mu)}$ are the local quantum states for subsystems a and b, respectively. Otherwise, it is called inseparable (entangled). Note that a classical bipartite probability distribution $p_{ab} = \{p_{ab}(i,j)\}$ is always separable in the above spirit since we can always find a probability vector $\{\lambda_\mu\}$ and families of probability distributions $p_a^{(\mu)}$ and $p_b^{(\mu)}$ for subsystems a and b, respectively, such that

$$p_{ab} = \sum_\mu \lambda_\mu p_a^{(\mu)} p_b^{(\mu)}.$$

For example, we may take the index $\mu = (\mu_1, \mu_2)$ which runs over the set $\{(i, j)\}$, and put $\lambda_\mu = p_{ab}(\mu_1, \mu_2)$, $p_a^{(\mu)}(i) = \delta_{\mu_1}(i)$, $p_b^{(\mu)}(j) = \delta_{\mu_2}(j)$, where δ is the Dirac delta function. Still another choice is to let the index μ run over the set $\{j\}$ and put $\lambda_\mu = p_b(\mu)$, $p_b^{(\mu)}(j) = \delta_\mu(j)$, $p_a^{(\mu)}(i) = p_{a|b}(i|\mu)$.

Although the conditional probability $p_{a|b}(\cdot|j)$ is always a bona fide probability distribution for any fixed j, and thus $p_{a|b}(i|j) \in [0, 1]$, the conditional density operators $\rho_{a|b}^-$ and $\rho_{a|b}^+$ are not necessarily bounded by $\mathbf{1}_{ab}$ (identity operator for the composite system) due to quantumness, that is, they may have an eigenvalue exceeding 1. If ρ_b is invertible, then apparently, $\rho_{a|b}^-$ and $\rho_{a|b}^+$ are bounded by $\mathbf{1}_{ab}$ if and only if it holds that

$$\mathbf{1}_a \otimes \rho_b \geqslant \rho_{ab}.$$

If ρ_b is not invertible, then we can make a small perturbation and use the continuity argument to establish the above equivalence. For example, if ρ_b is not invertible, we may put $\rho_{ab}^\epsilon = \epsilon \frac{\mathbf{1}_{ab}}{N} + (1-\epsilon)\rho_{ab}$ such that ρ_b^ϵ is invertible (here ϵ is a small positive number, and N is the dimension of the whole system). Then we can use the above equivalence and let $\epsilon \to 0$ to establish the equivalence in general cases. Note that $\mathbf{1}_a \otimes \rho_b \geqslant \rho_{ab}$ is precisely the reduction criterion for separability [8–11]. More specifically, if ρ_{ab} is separable, then it satisfies the above inequality, and accordingly, if it violates the above inequality, then it must be entangled. Therefore, the reduction criterion for entanglement has a simple probabilistic interpretation [12]: if the conditional density operators $\rho_{a|b}^-$ (or $\rho_{a|b}^+$) possess an eigenvalue exceeding 1 (a non-classical behavior), then ρ_{ab} is entangled.

It is of interest to study the relationships among the three conditional density operators $\rho_{a|b}$, $\rho_{a|b}^-$ and $\rho_{a|b}^+$. Clearly, all these operators are non-negative, and are not necessarily bounded by $\mathbf{1}_{ab}$.

Firstly, we note that if ρ_{ab} is a bipartite density operator acting on $C^d \otimes C^d$ with the reduced state ρ_b non-degenerate, then

$$\mathrm{tr}\rho_{a|b} \leqslant d, \qquad \mathrm{tr}\rho_{a|b}^- = d, \qquad \mathrm{tr}\rho_{a|b}^+ = d. \tag{15}$$

The first inequality follows from the celebrated Golden–Thompson inequality [14, 15], which states that $\mathrm{tr}\,\mathrm{e}^{X+Y} \leqslant \mathrm{tr}\,\mathrm{e}^X\mathrm{e}^Y$ for Hermitian matrices X and Y, and the strict inequality occurs whenever ρ_{ab} does not commute with $\mathbf{1}_a \otimes \rho_b$. The relation $\mathrm{tr}\rho_{a|b}^- = d$ follows from

$$\begin{aligned} \mathrm{tr}\rho_{a|b}^- &= \mathrm{tr}\rho_{ab}^{1/2}\big(\mathbf{1}_a \otimes \rho_b^{-1}\big)\rho_{ab}^{1/2} \\ &= \mathrm{tr}\big(\mathbf{1}_a \otimes \rho_b^{-1}\big)\rho_{ab} \qquad \text{(by the cyclic property of trace)} \\ &= \mathrm{tr}\big(\rho_b^{-1}\mathrm{tr}_a\rho_{ab}\big) \\ &= \mathrm{tr}\,\mathbf{1}_a = d, \end{aligned}$$

and the identity $\mathrm{tr}\rho_{a|b}^+ = d$ follows similarly. Furthermore, $\mathrm{tr}_a\rho_{a|b}^+ = \mathbf{1}_b$, which is a quantum analog for the classical identity $\sum_i p_{a|b}(i|j) = 1$ for any fixed j. A similar identity does not hold for $\rho_{a|b}$ and $\rho_{a|b}^-$ in general.

Secondly, by noting the fact that $\det(XY) = \det(X)\cdot\det(Y)$ for any positive definite matrices X and Y (here det denotes the determinant, which is equal to the product of eigenvalues of the relevant matrix), we obtain that

$$\mathrm{tr}\ln\rho_{a|b} = \mathrm{tr}\ln\rho_{a|b}^- = \mathrm{tr}\ln\rho_{a|b}^+$$

if the conditional density operators are non-degenerate.

Thirdly, noting the weak majorization relation $\mathrm{e}^{X+Y} \preceq_{\mathrm{w}} \mathrm{e}^{X/2}Y\mathrm{e}^{X/2}$ for any Hermitian matrices X and Y due to Thompson [15], first taking $X = \ln\rho_{ab}$, $Y = \ln\big(\mathbf{1}_a \otimes \rho_b^{-1}\big)$ and then taking $X = \ln\big(\mathbf{1}_a \otimes \rho_b^{-1}\big)$, $Y = \ln\rho_{ab}$, we have

$$\rho_{a|b} \preceq_{\mathrm{w}} \rho_{a|b}^-, \qquad \rho_{a|b} \preceq_{\mathrm{w}} \rho_{a|b}^+.$$

Recall that if $\{x_i\}$ and $\{y_i\}$ are the respective eigenvalues of X and Y enumerated decreasingly as $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$ and $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_n$, then $X \preceq_{\mathrm{w}} Y$ is defined as $\sum_{i=1}^{k} x_i \leqslant \sum_{i=1}^{k} y_i$ for $k = 1, 2, \ldots, n$. If in addition $\operatorname{tr} X = \operatorname{tr} Y$, then we say $X \preceq Y$.

Finally, suppose that $\rho_{a|b}$, $\rho^-_{a|b}$ and $\rho^+_{a|b}$ are non-degenerate, and thus positive definite, then we have the following majorization relations,

$$\ln \rho_{a|b} \preceq \ln \rho^-_{a|b}, \qquad \ln \rho_{a|b} \preceq \ln \rho^+_{a|b}, \tag{16}$$

which follow from the general result [16] (p 22)

$$\ln X + \ln Y \preceq \ln X^{1/2} Y X^{1/2}$$

for any positive definite matrices X and Y. In fact, putting $X = \rho_{ab}$ and $Y = \mathbf{1}_a \otimes \rho_b^{-1}$ we obtain the first relation in (16), and taking $X = \mathbf{1}_a \otimes \rho_b^{-1}$ and $Y = \rho_{ab}$ we obtain the second one. For the applications of majorization theory in the quantum information theory, see [17–19].

3. Quantum conditional entropies

Based on the conditional density operators $\rho^-_{a|b}$ and $\rho^+_{a|b}$ defined by equations (13) and (14), respectively, we may define two more quantum extensions of conditional entropy as

$$S^-(\rho_{ab}|\rho_b) = -\operatorname{tr}\rho_{ab} \ln \rho^-_{a|b} \tag{17}$$

and

$$S^+(\rho_{ab}|\rho_b) = -\operatorname{tr}\rho_{ab} \ln \rho^+_{a|b}, \tag{18}$$

which are both intuitive quantum analogs of the classical equation (3), and can also be interpreted as particular cases of generalized relative entropy functionals. We emphasize that in equations (17) and (18), the support of ρ_{ab} should be contained in the support of $\rho^-_{a|b}$ and $\rho^+_{a|b}$, respectively; otherwise, the conditional entropies are defined to be infinite.

A natural question arises: what are the relationships among the quantum conditional entropies $S(\rho_{ab}|\rho_b)$, $S^-(\rho_{ab}|\rho_b)$ and $S^+(\rho_{ab}|\rho_b)$?

Apparently, when $\mathbf{1}_a \otimes \rho_b$ commutes with ρ_{ab}, all the conditional density operators $\rho_{a|b}$, $\rho^-_{a|b}$ and $\rho^+_{a|b}$ coincide, and consequently, all the quantum conditional entropies are identical. Moreover, based on inequalities (15) and (16), it is tempting to guess that $S^-(\rho_{ab}|\rho_b) \leqslant S(\rho_{ab}|\rho_b)$ and $S^+(\rho_{ab}|\rho_b) \leqslant S(\rho_{ab}|\rho_b)$. Surprisingly, while the first is true, the second is reversed, namely, we have the following dominance relations:

$$S^-(\rho_{ab}|\rho_b) \leqslant S(\rho_{ab}|\rho_b) \leqslant S^+(\rho_{ab}|\rho_b). \tag{19}$$

The above intriguing inequalities follow from a general result of Hiai and Petz [20]: for any non-negative matrices X and Y and any $p > 0$, it holds that

$$p^{-1}\operatorname{tr} X \ln Y^{p/2} X^p Y^{p/2} \leqslant \operatorname{tr} X(\ln X + \ln Y) \leqslant p^{-1}\operatorname{tr} X \ln X^{p/2} Y^p X^{p/2}.$$

Namely, if we take $p = 1$, $X = \rho_{ab}$ and $Y = \mathbf{1}_a \otimes \rho_b^{-1}$ in the above inequality chain, we readily obtain (19) (note the minus sign in the definition of conditional entropies).

When talking about conditional entropies, we should emphasize that in the quantum case, they may take negative values, which is radically different from the classical case. This is intimately related to the fact that the conditional density operators may have eigenvalues larger than 1, as well as to the phenomena of entanglement. Some very deep and interesting informational interpretation of the negativity of the conditional entropy $S(\rho_{ab}|\rho_b)$ has been discovered recently by Horodecki *et al* [21, 22].

4. Measures of quantumness

Having introduced two alternative variants of quantum conditional entropies, we now use them to define two measures of quantumness of a bipartite quantum state. The differences

$$\mathcal{D}^{-}(\rho_{ab}) = S(\rho_{ab}|\rho_b) - S^{-}(\rho_{ab}|\rho_b) \tag{20}$$

and

$$\mathcal{D}^{+}(\rho_{ab}) = S^{+}(\rho_{ab}|\rho_b) - S(\rho_{ab}|\rho_b) \tag{21}$$

may be regarded as two measures characterizing certain quantum nature inherent in ρ_{ab} with respect to its marginal b. These quantities may arise not only from the entanglement, but also from the quantumness in separable states. These measures of the quantumness should be compared with the quantum discord defined by Olliver and Zurek via equation (10). They are different quantities, though with similar formal definitions.

For $\mathcal{D}(\rho_{ab}|\{B_j\})$ defined by equation (10), in order to get rid of the dependence on the measurement $\{B_j\}$, Zurek further defined the least quantum discord

$$\mathcal{D}(\rho_{ab}) = \inf_{\{B_j\}} \mathcal{D}(\rho_{ab}|\{B_j\})$$

and used it to study Maxwell's demons [23]. Due to the complicated minimization involved, this quantity, while theoretically fundamental, is hard to evaluate in practice. While the quantities $\mathcal{D}^{-}(\rho_{ab})$ and $\mathcal{D}^{+}(\rho_{ab})$ can be both straightforwardly calculated.

To gain some intuitive feeling on the magnitudes of the discord $\mathcal{D}^{-}(\rho_{ab})$ and $\mathcal{D}^{+}(\rho_{ab})$, we evaluate them for several examples of quantum states.

Example 1. Let ρ_{ab} be the Werner state acting on $C^d \otimes C^d$ defined as [13, 24, 25]

$$\rho_{ab} = \frac{d-\theta}{d^3-d}\mathbf{1}_{ab} + \frac{d\theta-1}{d^3-d}F, \qquad \theta \in [-1, 1].$$

Here, $F = \sum_{i,j=1}^{d} |ij\rangle\langle ji|$ is the flip operator with $\{|ij\rangle\}$ an orthonormal base of product states for the composite system. Recall that ρ_{ab} is separable if and only if $\theta \in [0, 1]$, independent of the dimension. Since the reduced state $\rho_b = \mathbf{1}_b/d$, all the three versions of quantum conditional entropies coincide, and thus both the quantum discord $\mathcal{D}^{-}(\rho_{ab})$ and $\mathcal{D}^{+}(\rho_{ab})$ are identically zero. Similarly, let

$$\rho_{ab} = \frac{1-\theta}{d^2-1}\mathbf{1}_{ab} + \frac{d^2\theta-1}{d^2-1}|\Psi^+\rangle\langle\Psi^+|, \qquad \theta \in [0, d]$$

be the isotropic state on $C^d \otimes C^d$, which is separable if and only if $\theta \in [0, 1]$ [24–26]. Here, $|\Psi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$ with $\{|i\rangle\}$ constituting an orthonormal base for C^d. Then $\rho_b = \mathbf{1}_b/d$ and consequently, we also have $\mathcal{D}^{-}(\rho_{ab}) = \mathcal{D}^{+}(\rho_{ab}) = 0$.

Example 2. In a two-qubit system, consider the following state (under standard base) with parameter $x \in (0, 1)$,

$$\rho_{ab} = \begin{pmatrix} \frac{1-x}{3} & 0 & 0 & 0 \\ 0 & \frac{x}{2} & \frac{x}{4} & 0 \\ 0 & \frac{x}{4} & \frac{x}{2} & 0 \\ 0 & 0 & 0 & \frac{2(1-x)}{3} \end{pmatrix}.$$

By the PPT criterion which is necessary and sufficient for 2×2 and 2×3 states [27], we know that the above state is entangled if and only if $x \in (0.6535, 1]$. The graphs of $\mathcal{D}^{-}(\rho_{ab})$ and $\mathcal{D}^{+}(\rho_{ab})$ versus the parameter x are depicted in figure 1. In this case, $\mathcal{D}^{-}(\rho_{ab}) \leqslant \mathcal{D}^{+}(\rho_{ab})$, and both are very small compared with the quantum conditional entropies.

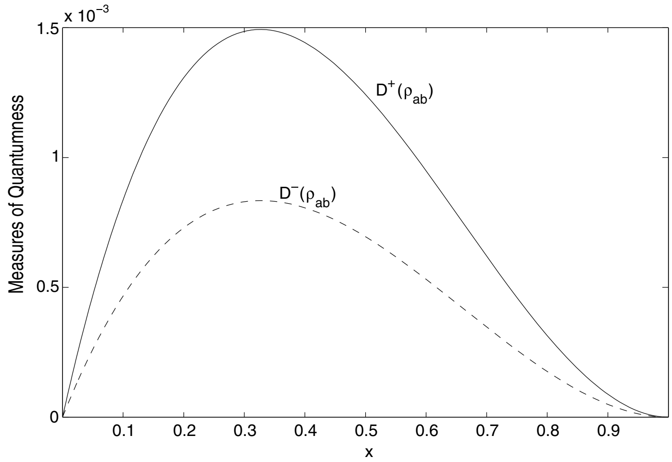

Figure 1. The graphs of $\mathcal{D}^-(\rho_{ab})$ and $\mathcal{D}^+(\rho_{ab})$ versus x for the state in example 2. We see that $\mathcal{D}^-(\rho_{ab}) \leqslant \mathcal{D}^+(\rho_{ab})$ in this case. Moreover, both $\mathcal{D}^-(\rho_{ab})$ and $\mathcal{D}^+(\rho_{ab})$ are very small relative to the conditional entropies.

Example 3. For $0 < x < 1$, consider the 1/2 fraction of the state ρ_x and 1/2 of the maximally mixed state $\mathbf{1}/9$ in $C^3 \otimes C^3$,

$$\rho_{ab} = \tfrac{1}{2}\left(\rho_x + \tfrac{\mathbf{1}}{9}\right)$$

where

$$\rho_x = \frac{1}{8x+1}\begin{pmatrix} x & 0 & 0 & 0 & x & 0 & 0 & 0 & x \\ 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & x & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+x}{2} & 0 & \frac{\sqrt{1-x^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 \\ x & 0 & 0 & 0 & x & 0 & \frac{\sqrt{1-x^2}}{2} & 0 & \frac{1+x}{2} \end{pmatrix}$$

is the remarkable state introduced by Horodecki [27].

The graphs of $\mathcal{D}^-(\rho_{ab})$ and $\mathcal{D}^+(\rho_{ab})$ versus parameter x are depicted in figure 2. In this case, $\mathcal{D}^-(\rho_{ab}) \geqslant \mathcal{D}^+(\rho_{ab})$.

Example 4. Consider the two-qubit entangled pure state with parameter $x \in (0, 1)$,

$$\rho_{ab} = |\Psi_x\rangle\langle\Psi_x|, \qquad \text{with} \qquad |\Psi_x\rangle = \sqrt{x}|01\rangle + \sqrt{1-x}|10\rangle.$$

In the standard base, ρ_{ab} has the matrix form

$$\rho_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & x & \sqrt{x(1-x)} & 0 \\ 0 & \sqrt{x(1-x)} & 1-x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

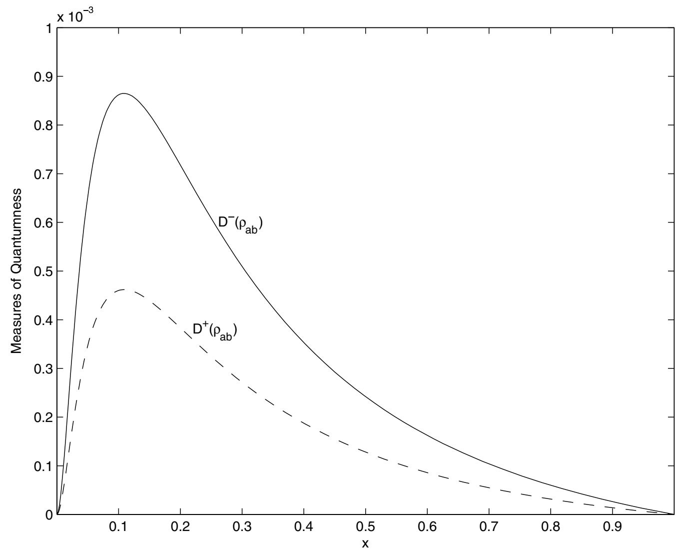

Figure 2. The graphs of $\mathcal{D}^-(\rho_{ab})$ and $\mathcal{D}^+(\rho_{ab})$ versus x for the state in example 3. We see that $\mathcal{D}^+(\rho_{ab}) \leqslant \mathcal{D}^-(\rho_{ab})$ in this case.

with the marginal state

$$\rho_b = \begin{pmatrix} 1-x & 0 \\ 0 & x \end{pmatrix}$$

for subsystem b.

The conditional entropy $S(\rho_{ab}|\rho_b)$ is readily evaluated as

$$S(\rho_{ab}|\rho_b) = S(\rho_{ab}) - S(\rho_b) = (1-x)\ln(1-x) + x\ln x.$$

Noting that $\rho_{ab}^{1/2} = \rho_{ab}$, the conditional density operator $\rho_{a|b}^-$ can be easily evaluated as

$$\rho_{a|b}^- = \rho_{ab}^{1/2}\big(\mathbf{1}_a \otimes \rho_b^{-1}\big)\rho_{ab}^{1/2} = 2|\Psi_x\rangle\langle\Psi_x| = 2\rho_{ab}.$$

Consequently,

$$S^-(\rho_{ab}|\rho_b) = -\mathrm{tr}\rho_{ab}\ln\rho_{a|b} = -\ln 2$$

which is a constant independent of parameter x.

Now we evaluate the conditional density operator $\rho_{a|b}^+$,

$$\rho_{a|b}^+ = \big(\mathbf{1}_a \otimes \rho_b^{-1/2}\big)\rho_{ab}\big(\mathbf{1}_a \otimes \rho_b^{-1/2}\big) = 2|\Psi_{1/2}\rangle\langle\Psi_{1/2}|,$$

which is independent of parameter x. When $x \neq 1/2$, since the support of ρ_{ab} is the one-dimensional space spanned by $|\Psi_x\rangle$, which is not included in the support of $\rho_{a|b}^+$ spanned by $|\Psi_{1/2}\rangle$, the conditional entropy $S^+(\rho_{ab}|\rho_b) = \infty$ by definition.

Thus, for this example, when $x = 1/2$, both $\mathcal{D}^-(\rho_{ab})$ and $\mathcal{D}^+(\rho_{ab})$ vanish, while for $x \neq 1/2$, we have

$$\mathcal{D}^-(\rho_{ab}) = \ln 2 + (1-x)\ln(1-x) + x\ln x, \qquad \mathcal{D}^+(\rho_{ab}) = \infty.$$

From the above four examples, we see that the measures of quantumness may range from 0 to ∞, and $\mathcal{D}^+(\rho_{ab})$ may be either larger or smaller than $\mathcal{D}^-(\rho_{ab})$.

5. Conclusion

Motivated by the classical notions of conditional probability and conditional entropy, we have compared several quantum extensions of conditional probability and used them to define two alternative variants of quantum conditional entropies. We have emphasized the fundamental difference between the conditional density operators $\rho_{a|b}^{-}$, $\rho_{a|b}^{+}$ and $\rho_{ab}(j)$. The latter depends on the choice of measurement performed over subsystem b, while both $\rho_{a|b}^{-}$ and $\rho_{a|b}^{+}$ are intrinsic. The notion of conditional density operators and their non-classical behaviors also provide an intuitive interpretation of the reduction criterion for separability of quantum states.

The two alternative variants of quantum conditional entropies, together with the conventional one, are used to exhibit the quantumness of a bipartite state in the spirit of Olliver and Zurek from a new perspective. Consequently, we have three natural measures of quantumness; they are $\mathcal{D}(\rho_{ab}|\{\Pi_j\})$, $\mathcal{D}^{-}(\rho_{ab})$ and $\mathcal{D}^{+}(\rho_{ab})$ defined by equations (10), (20) and (21), respectively. The latter two vanish whenever the reduced density operator ρ_b commutes with ρ_{ab}. We can also develop a completely similar approach using the reduced density ρ_a, rather than ρ_b, as a condition.

Because the nature of quantumness is highly subtle, and no single quantity can capture all its features, the combination of various measures of quantumness shed more light on the nature of quantum states.

Acknowledgments

The work is supported by NSFC, Grant no 10571166, and by the Science Fund for Creative Research Groups. The authors are very grateful to the referee for constructive suggestions.

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